Statistical Mechanics of 2+1 Gravity From Riemann Zeta Function and Alexander Polynomial: Exact Results

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Abstract

In the recent publication (Journal of Geometry and Physics, 33 (2000) 23-102) we have demonstrated that dynamics of 2+1 gravity can be described in terms of train tracks. Train tracks were introduced by Thurston in connection with description of dynamics of surface automorphisms. In this work we provide an example of utilization of general formalism developed earlier. The complete exact solution of the model problem describing equilibrium dynamics of train tracks on the punctured torus is obtained. Being guided by similarities between the dynamics of 2d liquid crystals and 2+1 gravity the partition function for gravity is mapped into that for the Farey spin chain. The Farey spin chain partition function, fortunately, is known exactly and has been thoroughly investigated recently. Accordingly, the transition between the pseudo-Anosov and the periodic dynamic regime (in Thurston's terminology) in the case of gravity is being reinterpreted in terms of phase transitions in the Farey spin chain whose partition function is just the ratio of two Riemann zeta functions. The mapping into the spin chain is facilitated by recognition of a special role of the Alexander polynomial for knots/links in study of dynamics of self homeomorphisms of surfaces. At the end of paper, using some facts from the theory of arithmetic hyperbolic 3-manifolds (initiated by Bianchi in 1892), we develop systematic extension of the obtained results to noncompact Riemann surfaces of higher genus. Some of the obtained results are also useful for 3+1 gravity. In particular, using the theorem of Margulis, we provide new reasons for the black hole existence in the Universe: black holes make our Universe arithmetic. That is the discrete Lie groups of motion are arithmetic.

1 Introduction and summary

The Riemann zeta function $\zeta(\beta)$ has been an object of intensive study in both mathematics [1-3] and physics [4,5] for quite some time.The reason for physicists interest in this function can be easily understood if one writes it in the form of a partition function Z(β) given by

$$\zeta(\beta) \equiv Z(\beta) = \sum_{n=1}^{\infty} \exp\{-\beta \ln n\}. \tag{1.1}$$

If β is interpreted as the inverse temperature then, naturally, questions arise:

a) what is the explicit form of the quantum mechanical Hamiltonian whose eigenvalues E_n are given by $E_n = \ln n$?

b)can such system undergo phase transition(s) if one varies the temperature? The goal of providing answers to both questions is at the forefront of current research activities both in physics [4,5] and mathematics [6]. Answers to these questions are being sought in connection with theories of random matrices and quantum chaos [4,5], non-commutative geometry [6] and Yang-Lee zeros [7]. According to the theory of Yang and and Lee the problem of existence of phase transitions can be reduced to the problem of existence of zeros of the partition function in the complex z-plane (where z may be related to either fugacity or the magnetic field, etc.). In the case of $Z(\beta)$, Eq.(1.1), one is also looking at analytic behavior of the partition function in the complex $\beta-$plane. Riemann had conjectured that,

$$Z(12 + it_m) = 0, \tag{1.2}$$

provided that $Re\ t_m \neq 0$ for all integer m's. This conjecture is known in the literature as Riemann hypothesis. Stated differently, the Riemann hypothesis is equivalent to the statement that all "nontrivial" zeros of the partition function Z(β) are located at the critical line $Re\beta = 12$. The "trivial" zeros are known [1-3] to be located at $\beta = -2, -4, -6, ...$,that is,

$$Z(\beta = -2m) = 0,$$
$$Z(\beta = -2m + 1) = -\mathrm{B}_{2m} 2m$$

for m=1,2,... and B_{2m} being the Bernoulli numbers.

Combinations of the Riemann zeta functions are also of physical interest. In particular, in this paper we shall be concerned with the following combination

$$\hat{Z}(\beta) = \zeta(\beta - 1)\zeta(\beta) = \sum_{n=1}^{\infty} \phi(n) n^{-\beta} \tag{1.4}$$

where $\phi(n)$ is the Euler totient function,

$$\phi(n) = n(1 - 1p_1) \cdots (1 - 1p_r), \tag{1.5}$$

which is just the number of numbers less than n and prime to n, provided that $n = p_1^{m_1} \cdots p_r^{m_r}$, and p_1, etc. are primes with respect to n. This partition function had appeared in mathematical physics literature in connection with the partition function for the number-theoretic spin chain [8-10] and in connection with calculations of the scattering S-matrix for the "leaky torus" quantum mechanical problem [11-13]. Remarkably enough, the results for the number-theoretic spin chain can be obtained as well from earlier works on mode locking and circle maps [14,15], e.g. see in particular Eq.(30) of Ref.[15], as the authors of Ref.[10] acknowledge. This fact is not totally coincidental as we shall explain below (in sections 2 and 3) and has been already anticipated based on our earlier works [16,17] on dynamics of 2+1 gravity.

In this paper we would like to demonstrate that the partition function, Eq.(1.4), can adequately describe statistical mechanics of Einsteinian 2+1 gravity if the underlying surface is the punctured torus. The restriction to the punctured torus case is not too severe and is motivated mainly by illustrative purposes: recall, that both the Seifert surfaces of the figure eight and the trefoil knots are just punctured toruses [18]. This observation allows us to make an easy connection between the dynamics of surface self homeomorphisms and the associated with its time evolution 3-manifolds which fiber over the circle [17] . These manifolds are just complements of the figure eight and trefoil knots in S^3 respectively.The Seifert surfaces of more complicated knots may naturally be of higher genus but, since both the trefoil and the figure knots belong to the category of fibered knots, only those knots and links which are fibered and the associated with them Seifert surfaces are relevant to the dynamics of 2+1 gravity [17]. The 3- manifolds associated with the figure eight and the trefoil knots are fundamentally different: the first one is known to belong to the simplest representative of the hyperbolic manifolds while the second corresponds to the so called Seifert fibered manifolds [19].The surface dynamics associated with the first is associated with pseudo-Anosov type of surface self homeomorphisms while the second one is associated with periodic self homeomorphisms. Both types of 3-manifolds are topologically very interesting and potentially contain wealth of useful physical information. In this work we only initiate their study with hope of returning to this subject in future publications.

In order for the reader to keep focus primarily on physical aspects of the problem, we feel, that some simple explanation of what follows is appropriate at this point. To avoid repetitions, we expect that our readers have some background knowledge of the results presented in our earlier published papers [16,17]. In particular, to help our intuition, we would like to exploit the fact that statics and dynamics of 2+1 gravity is isomorphic with statics and dynamics of textures in two dimensional liquid crystals. According to the existing literature on liquid crystals, e.g. [20], the liquid crystalline state can be found in several phases which physicists classify as liquid, solid, hexatic and gas. In mathematical literature the textures, e.g. like those in liquid crystals, are known as foliations [19,21]. Dynamics of textures is known accordingly as dynamics of foliations. Some of these foliations may contain singularities.These singularities are mistakenly being treated as Coulombic charges (while in 2+1 gravity is is well documented [17,22] that these singularities do not interact) since the **nonorientable** line fields are being confused with the **orientable** vector fields. The phase transitions in two dimensional liquid crystals are described in terms of the phase transitions in 2 dimensional Coulomb gas. These are known as the Kosterlitz-Thouless type transitions [23]. Although mathematically such explanation of phase transition is not satisfactory, nevertheless, one has to respect the experimental data associated with the liquid crystalline phases.The extent to which the Kosterlitz-Thouless interprepretation of phase transition is appropriate is discussed in the Appendix A.1. where it is being argued that, by analogy with transitions in the liquid helium (where the Kosterlitz-Thouless interpretation is normally used) the dynamical phase transition in 2+1 gravity resemble to some extent the Bose gas condensation type of transition.This analogy is incomplete, however, and is only being used for the sake of comparison with the existing literature. As Remark 4.1. indicates, the partition function of 2+1 gravity **without any appriximations** can be recast into the Lee-Yang form [7]. The calculations associated with such form require knowledge of distribution of zeros of the Riemann Zeta function and, hence, effectively, the proof of the Riemann hypothesis. Since this proof is not yet available (see, however, Ref.[5]) we employ alternative methods associated with recently developed thermodynamic/statistical mechanic formalism for description of phase transitions in the number-theoretic Farey spin chains [8-10]. To prove that dynamical transitions in gravity can be described in terms of transitions in the Farey spin chains several steps are required. In our earlier works [16,17] we have demonstrated that dynamic of 2+1 gravity is best described in terms of dynamic of train tracks. In section 2 we demonstrate how dynamic of train tracks can be mapped into dynamic of geodesic laminations. In turn, the dynamic of geodesic laminations is reformulated in terms of the sequence of Dehn twists. This sequence is actually responsible for the **dynamic in the Teichmüller space** of the punctured torus. Such dynamic is subject to the number-theoretic constraints associated with the Markov triples.The Markov triples had been known in physics literature for a while in connection with the trace maps [24] used for description of quasicrystals, 1d tight band Scrödinger equations, etc. In this work the Markov triples play a somewhat different role: they make the set of closed nonperipheral geodesics on the punctured torus discrete. Mathematically, the sequence of Dehn twists is written in terms of the product of the "right" and the "left" 2×2 Dehn matrices. The modulus of eigenvalues(the stretch factors) associated with such matrix product can be either greater than one or equal to one. In the first case one is dealing with the pseudo-Anosov and in the second, with the periodic (Seifert-fibered) dynamical regime. The results of section 2 (for the figure 8 and the trefoil knots) acquire new meaning in section 3 where they are reobtained with help of the associated Alexander polynomials.The stretch factors of section 2 are reobtained as zeros of the related Alexander polynomials. In the same section 3 we discuss the fiber bundle construction of 3-manifolds complementary to the figure eight and the trefoil knots in S^3. The sequence of Dehn twists, discussed in section 2, in this section is being associated with the operation of Dehn surgery (Dehn filing) performed on the 3-manifold related to the figure eight knot.The stretch factors produced as result of such surgery are reobtained with help of the Mahler measures which allow us to reinterpret these factors in terms of the topological entropies. Introduction of the Mahler measures, in addition, allows us to make direct connection between the dynamical phase transitions and the thermodynamic transitions in the sense of Yang and Lee [7]: zeros of the Alexander polynomial play similar role in dynamics as Yang-Lee zeros in thermodynamics. In section 4 we provide direct connection between the results of section 3 just described and the statistical mechanics formalism developed for the number-theoretic Farey spin chains [8-10]. With such connection established, dynamical phase transitions in 2 +1 gravity can be treated with formalism which is more familiar to physicists. Unfortunately, this more familiar formalism is applicable (at least at the present level of our understanding) only to the case of punctured torus. Fortunately, the final results obtained with help of such type of formalism can be reobtained in several different and independent ways.We discuss these alternative ways at the end of section 4 and in section 5 which is entirely devoted to development and refinements of these alternatives. The major reason of doing this lies in the opportunity of extension of the punctured torus results to the noncompact Riemann surfaces of **any** genus. This is accomplished using some results from the scattering theory for Poincare' (Eisenstein) series acting on 3-manifolds. These series had been recently discussed in our earlier work, Ref.[25], to which we refer for more details. The key result of section 4, the partition function for the case of punctured torus, happens to coincide with the scattering S-matrix (up to unimportant constant) obtained for 3-manifolds with one $\mathbf{Z}\oplus\mathbf{Z}$ cusp [26]. 3-manifolds containing multiple $\mathbf{Z}\oplus\mathbf{Z}$ cusps are associated with fibered links (as explained in section 3 and the Appendix A.3). The S-matrix for this case has been also obtained recently [27]. The determinant of this matrix produces the desired exact partition function for 2+1 gravity. In addition, the formalism allows us to obtain the volumes of the associated 3-manifolds exactly. Extension of the punctured torus results to surfaces of higher genus requires some careful analysis of the nontrivial mathematical problem of circle packing. This fact has some profound impact at development of the whole formalism. It happens, that such type of problems had been comprehensively studied by Bianchi already in 1892 [28] who championed study of the arithmetic hyperbolic manifolds. The notion of arithmeticity is rather involved. Since in physics literature (to our knowlege) it did not find its place yet, in Appendices A.2 and A.3 we supply the essentials needed for the uninterrupted reading of the main text. Surely, selection of the material in these appendices is subjective. But, it is hoped, that interested reader will be able to restore the missing details if it is required. The arithmeticity of 3-manifolds associated with 2+1 gravity stems from some very deep results of Riley [29] and Margulis [30] which we discuss to some extent in

Appendix A.3. The arithmeticity leads to some restrictions on groups of motions in symmetric spaces (e.g. hyperbolic space is symmetric space). Thanks to the Margulis Theorem A.3.11. and some results of Helgason [31] and Besse [32], the notion of arithmeticity is extendable to 3+1 gravity as well. In the case of 2+1 gravity we demonstrate, that the very existence of black holes makes such 2+1 Universe arithmetic. Since most of the Einstein spaces happen to be symmetric, we expect that they are arithmetic in addition in view of the Margulis theorem [30].This possibility is realized in nature only if the black holes exist in our Universe. The black holes make our Universe arithmetic.

2 From train tracks to geodesic laminations

2.1 Dynamics of train tracks on punctured torus

Dynamics of pseudo-Anosov homeomorphisms on the four punctured sphere was studied in some detail in Ref.[33].Closely related but more physically interesting case is associated with study of self homeomorphisms of the punctured torus.The Poincare-Hopf index theorem requires existence of two Y-type singularities, each having index -12, as it is explained in Ref.[16]. These singularities can move on the surface of the punctured torus thus giving rise to the train tracks dynamics as depicted in Fig.1**.**

From this picture it follows, that the nontrivial dynamics is effectively caused by sequence of meridional τ_m and longitudinal τ_l Dehn twists. Using the rules set up for dynamics of train tracks [16], one can easily calculate the transition matrix by noticing that topologically the state h) is the same as a) while the weights on the branches are different. This allows us to write the following system of equations

$$a' = b + 2a,$$

$$b' = c, \tag{2.1}$$

$$c' = a + 2c.$$

These results can be neatly presented in the matrix form :

$$= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} .(2.2)$$

Figure 1: Fragment of the train track dynamics on once punctured torus

$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The incidence matrix just obtained can be found in Penner's article, Ref.[34], where it was presented without derivation.The largest eigenvalue λ is found to be

$$\lambda = 12(3 + \sqrt{5}). \tag{2.3}$$

Since $\lambda > 1$,this indicates that the dynamics depicted in Fig.1 is of pseudo-Anosov type. We shall reobtain this result for λ below, e.g. see Eq.(2.16), and in section 3 using totally different methods. In fact, the presentation above is only given for the sake of comparison with these new methods to be discussed in section 3. These new methods provide the most natural connections between the dynamics of 2+1 gravity and the theory of knots and 3-manifolds.

The reminder of this section is devoted to exposition of some mathematical results which will be used in the rest of this paper.

2.2 The Markov triples

To begin, let us recall [16,17,19], that a *geodesic lamination* on a hyperbolic surface S is a closed subset of S made of union of disjoint simple geodesics. When lifted to the universal cover, i.e. to the Poincare$'$ disc $\mathcal{D}$ whose boundary is a circle S^1_∞ at infinity, the endpoints of the geodesic lamination determine a closed subset (actually, a Mobius strip)

$$\mathcal{E} = (S^1_\infty \times S^1_\infty - \Delta)/Z_2 \tag{2.4}$$

where Δ is diagonal $(x, x), x \in S^1_\infty$ and the factor Z_2 reflects the fact that the circle segments representing these geodesics are unoriented, that is the picture remains unchanged if the ends of each geodesic which lie on S^1_∞ are interchanged. This fact has been discussed and used already in our earlier work, Ref.[16]. Since, by definition, the lamination is made of disjoint set of geodesics, when lifted to $\mathcal{D}$, there are no circle segments (representing geodesics) which intersect with each other. It is intuitively clear, that the dynamics of train tracks should affect the dynamics of geodesics. We would like to make this intuitive statement more precise. To this purpose, we notice that, when lifted to the universal cover, this dynamics causes some homeo(diffeo)morphisms of the circle S^1_∞ thus making clear the connections with circle maps and mode locking [14,15].

If $G = \pi_1(S)$ is the fundamental group of surface S, the endpoint subset $\mathcal{E}$ remains invariant under the action of $\pi_1(S)$. Suppose, that the surface S has boundaries, e.g. a hole in the case of a torus. Let $P \sqsubseteq G$ be the set of *peripheral*